Indian Statistical Institute

Midterm Examination 2018-2019

Analysis II, B.Math First Year

Time : 3 Hours   Date : 25.02.2019   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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Note: *(i) Answer **all** questions. (ii)* (X, d) *and* (Y, d) *are metric spaces. (iii)* $B_r(x) = \{y \in X : d(x, y) < r\}$. *(iv)* $P(X)$ = *power set of* X. *(v)* $\#X$ = *cardinality of* X.

Q1. (10 marks) Let U be an open subset of X, and let $x \in U$. Is $U \setminus \{x\}$ open in X? Justify your answer.

Q2. (10 marks) Let $f, g : X \to \mathbb{R}_u$ be continuous functions, and let $C = \{x \in X : f(x) = g(x)\}$. Prove that C is closed.

Q3. (10 marks) Prove that a discrete metric space is complete.

Q4. (10 marks) Define $C = \{x \in X : B_r(x)$ is uncountable for every $r > 0\}$. Prove that C is a closed set.

Q5. (15 marks) Let $\tilde{d}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ for all $A, B \subseteq X$. State whether the following statements are true or false. Justify your answer.
(i) If $A \cap B \neq \emptyset$, then $\tilde{d}(A, B) = 0$. (ii) If $\tilde{d}(A, B) = 0$, then $A \cap B \neq \emptyset$. (iii) $(P(X), \tilde{d})$ is a metric space.

Q6. (15 marks) Let D be a dense subset of X such that every Cauchy sequence in D converges in X. Prove that X is complete.

Q7. (15 marks) Prove that $\mathbb{R}^n_u \setminus \mathbb{Q}^n$, $n > 1$, is connected.

Q8. (15 marks) Let X be a countable metric space, and let $\#X \geq 1$. Prove that X is connected if and only if $\#X = 1$.

Q9. (15 marks) Let $f : X \to Y$ be a continuous function, and let

$$Graph(f) = \{(x, f(x)) : x \in X\}.$$

Prove that $Graph(f)$, with respect to the metric inherited from the product metric, is homeomorphic to X.